Problem Set 8

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 8 covers materials from §5.3 – §5.5.

1. Let f and g be two differentiable functions, whose derivatives f' and g' are both continuous. The following table shows some values of f, g, f' and g'.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	0	2	-3
2	0	3	0	-1
4	1	6	1	1

Using the information from this table, evaluate the following integrals.

(a) $\displaystyle\int_0^2 [3f'(x) - 4xg'(x^2)]dx$

(b) $\displaystyle\int_{-1}^1 [f(x+3)]^3 f'(x+3)dx$

2. Evaluate the integral

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}\,dx$$

using the substitution $u = \frac{1}{x}$.

3. (a) Let $a > 0$ and let $f: [0, a] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

(b) Using the result from (a), evaluate the following integrals.

(i) $\displaystyle\int_0^{\frac{\pi}{2}} (\cos x - \sin x)^{1013} dx$

(ii) $\displaystyle\int_0^{\pi} x \sin^2 x\, dx$

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x + \cos x}\,dx$

4. Evaluate the integral

$$\int_0^{\frac{1}{2}} \frac{t^2 + 2t + 3}{\sqrt{1-t^2}}\,dt.$$

Hint: The integral $\int_0^{1/2} \sqrt{1-t^2}\, dt$ can be computed using simple geometry.

5. (a) Let $a > 0$ and let $f: [0, a] \to \mathbb{R}$ be a continuous function.

   (i) Prove that
   $$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

   (ii) If there exists a real constant c such that
   $$f(x) + f(a-x) = c \qquad \text{for every } x \in [0, a],$$
   show that
   $$\int_0^a f(x)dx = af\left(\frac{a}{2}\right).$$

   (b) Using the result from (a), evaluate the integral
   $$\int_0^{2\pi} \frac{1}{e^{\sin^3 x} + 1} dx.$$

6. Evaluate the following antiderivatives or integrals which involve trigonometric functions.

   (a) $\int \sin^4 x \, dx$

   (b) $\int_{-\pi}^{\pi} \cos 1013x \cos 2025x \, dx$

   (c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\sin 3x}{\sin x} + \frac{\cos 3x}{\cos x}\right) dx$ *Hint*: Recall Q17(c) of Problem Set 1.

   (d) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$

   (e) $\int \sec^4 x \tan^2 x \, dx$

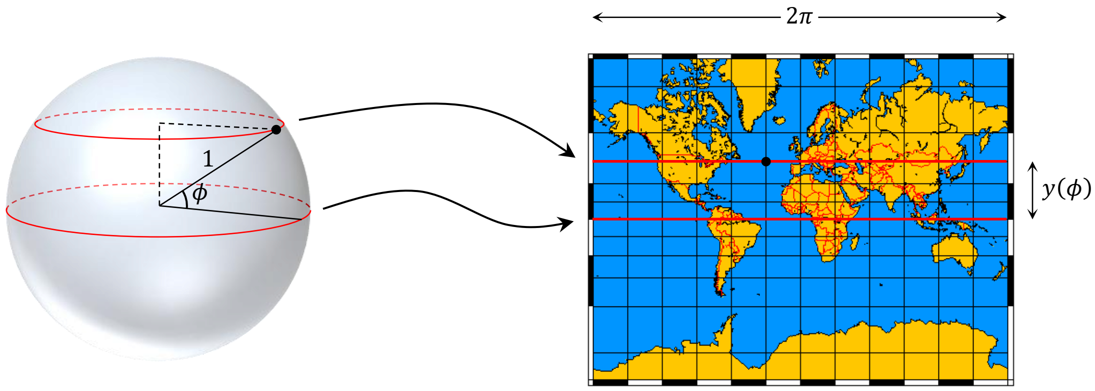
7. Household electricity in Hong Kong is supplied in the form of alternating current with voltage varying from $311\,\mathrm{V}$ to $-311\,\mathrm{V}$ and frequency 50 cycles per second ($50\,\mathrm{Hz}$). Thus the voltage E (in V) as a function of time t (in seconds) is
   $$E(t) = 311 \sin(100\pi t).$$

   (a) The ***mean value*** of a function f over an interval $[a, b]$ is defined by its integral over $[a, b]$ divided by the length of $[a, b]$, i.e.
   $$\frac{1}{b-a} \int_a^b f(t)dt.$$
   Calculate the mean value of the voltage function E over a time interval of 1 second.

   (b) The ***root-mean-squared value*** (or ***RMS value***) of a function f over an interval $[a, b]$ is defined by the square root of the mean value of the function f^2 over $[a, b]$. Calculate the RMS value of the voltage function E over a time interval of 1 second.

   (c) In some other places, the household electricity supplied often has an RMS value of $110\,\mathrm{V}$ instead. What is the amplitude of the voltage function that is necessary in these places?

8. Assume that the Earth surface is in the shape of a sphere with radius 1. In geography, each point on the Earth surface is described by a pair of angles called the latitude and the longitude. In particular, the ***latitude*** $\phi \in [-\pi/2, \pi/2]$ (or $\phi \in [-90°, 90°]$) of a point measures the angle from the Earth's center between the point and the equator (see the left diagram below).



Mercator projection is a popular method of constructing a rectangular world map. The method is designed so that all angle measurements on the Earth surface are preserved in the map. Now suppose that every circle on the Earth with a fixed latitude is "unrolled" to become a horizontal line segment of the same length 2π in the map (see the right diagram).

(a) Consider the circle on the Earth surface with latitude ϕ. Show that its "unrolled" length in the map is stretched by a factor of $\sec\phi$.

(b) Consider a point on the Earth surface with latitude ϕ, and let $y(\phi)$ be its (vertical) displacement from the equator in the map ($y(\phi) > 0$ in the northern hemisphere and $y(\phi) < 0$ in the southern hemisphere). In order to preserve all angle measurements, it can be shown that this function $y(\phi)$ must satisfy the differential equation
$$y'(\phi) = \sec\phi\,.$$
Now find the function $y(\phi)$.

(c) Suppose that the map is in the shape of a square, i.e. its length and its width are both 2π. If the equator is still placed at the center of the map as usual, what is the largest angle α so that the square-shaped map covers all points on the Earth with latitude $\phi \in [-\alpha, \alpha]$? Express the size of α in degrees.

9. Evaluate each of the following, using antidifferentiation or integration by parts if necessary.

(a) $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^{\pi} (x + \sin x)^2 dx$

(c) $\displaystyle\int x^2 \ln(1 + x)\,dx$

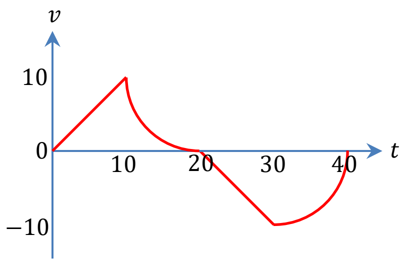
(d) $\displaystyle\int_1^{e} \sin(\ln x)\,dx$

10. It is estimated that the daily sales for a certain item is given by

$$S(t) = 100t^2 e^{-\frac{t}{5}},$$

where t is the number of days starting from now. What is the estimated total sales for the item over the first 100 days?

11. The diagram below shows the velocity v (in $\mathrm{m/s}$) of a vehicle at time t (in seconds) during a 40-second journey along a straight line. The curved portions of the graph are quarter-circles.



(a) What is the final displacement of the vehicle from its initial position?

(b) What is the furthest displacement of the vehicle from its initial position throughout the whole journey?

(c) What is the total distance traveled by the vehicle during the whole journey?